Recall

**THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form)** Let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M$ and $N$ having continuous first partial derivatives in an open region containing $R$. Then the counterclockwise circulation of $\mathbf{F}$ around $C$ equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over $R$.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\,dx + N\,dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Curl integral}} \tag{3}$$

Let $\vec{F}' = -N\vec{i} + M\vec{j}$. Then $\vec{F}' \cdot \vec{T} = (-N\vec{i} + M\vec{j}) \cdot (\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j})$

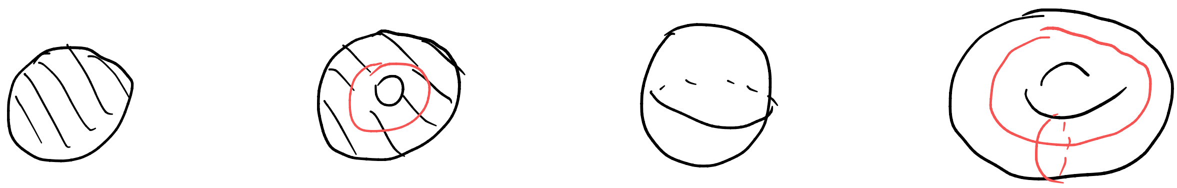
$= M dy - N dx$

$= \vec{F} \cdot \vec{n}$

**THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form)** Let $C$ be a piecewise smooth, simple closed curve enclosing a region $R$ in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with $M$ and $N$ having continuous first partial derivatives in an open region containing $R$. Then the outward flux of $\mathbf{F}$ across $C$ equals the double integral of div $\mathbf{F}$ over the region $R$ enclosed by $C$.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}} \tag{4}$$

"Def" A connected domain D is called <u>simply-connected</u> if every closed curve in D can be contracted to a pt

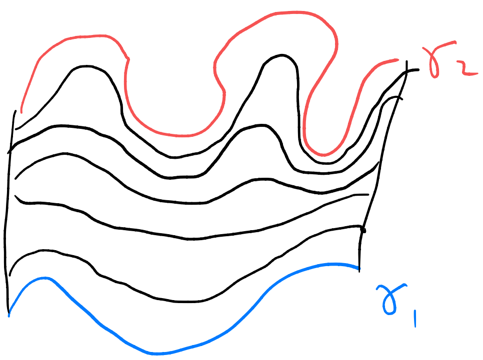


✓ X ✓ X

Def Let $\gamma: S^1 \to D$ be a loop. If $\exists$ a continuous map $H: S^1 \times I \to D$ s.t. $H(x,0) = \gamma(x)$ and $H(x,1) \equiv pt$. Then we say $\gamma$ can be contracted to a pt.

More generally, Let $\gamma_1, \gamma_2 : I \to D$ be two curves. We say $\gamma_1$ and $\gamma_2$ are <u>homotopic</u> if $\exists$ a continuous map $H: I \times I \to D$ s.t. $H(x,0) = \gamma_1(x)$ and $H(x,1) = \gamma_2(x)$.

Denoted by $\gamma_1 \simeq \gamma_2$. $H$ is called a <u>homotopy</u> between $\gamma_1$ and $\gamma_2$




Remark: Equivalently, we say $D$ is <u>simply connected</u> if every loop in $D$ is homotopic to a constant map.

Remark: Many statements only hold on simply connected spaces.

i.e. component test for exact form, conservative vector field.

e.g. $\vec{F} = \frac{-y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j} + 0\vec{k}$

Line integral of exact forms (or holomorphic functions) on a simply connected space is independent of the choice of paths.